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Discrete Dynamical Equations in Minkowski Space

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Discrete difference equations in Minkowski space are obtained and the discrete Minkowski force is shown to be a four-vector. A transformation from a discrete dynamical equation in Minkowski space to a Lorentz-invariant difference equation in one-dimensional space is given.

1. INTRODUCTION

The development of modern digital computers has led to special approaches to many mathematical and physical problems. The replacement of derivatives by differences is a common method for the solution of problems in mechanics. Discrete Newtonian and relativistic mechanics have been developed especially by D. Greenspan (1974, 1976, 1977). He presents, among other things, a purely arithmetical approach to elements of special relativity theory (1976, 1977).

As is well known, in continuous special relativity the basic equation

$$F=\frac{c^2 m}{c^2-v^2}\frac{dv}{dt} \tag{1}$$

is not invariant under Lorentz transformation in more than one-dimensional space. The analogous situation takes place with adequate discrete equation given by D. Greenspan (1976, 1977). In connection with this, the dynamical relativistic equations are formulated in Minkowski space (Aharoni, 1959; Bergman, 1942; Muirhead, 1973; Möller, 1973; Schwartz, 1968).

In this paper, a class of discrete dynamical equations in Minkowski space is obtained. Moreover, it is proved that the discrete Minkowski force

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is, as in the continuous case, a four-vector. Finally, on the basis of discrete mechanics, the transformation from a difference relativistic equation in Minkowski space to the above-mentioned Lorentz-invariant Greenspan equation in one-dimensional space is presented.

2. BASIC CONCEPTS

Let us assume that the location of particle P at moment t_k is given by the rectangular coordinates $x_{lk}=x_l(t_k)$; $k=0,1,2,\ldots$; $l=1,2,3$; and let the particle P have velocity $v_k=[v_{1k},v_{2k},v_{3k}]$. Let velocity and location be related by the following difference equations:

$$v_{lk}=\frac{\Delta x_{lk}}{\Delta t_k} \tag{2}$$

where the symbol Δ represents a forward difference operator defined by

$$\Delta A_k=A_{k+1}-A_k \tag{3}$$

for arbitrary quantity A. Moreover, let us define γ_k by

$$\gamma_k=\left(1-v_k^2/c^2\right)^{1/2} \tag{4}$$

where $v_k^2=v_{1k}^2+v_{2k}^2+v_{3k}^2$ and c denotes the velocity of light. In analogy to the continuous case (Bergman, 1942; Aharoni, 1959) we define the proper time by

$$\delta\tau_k=\gamma_k\Delta t_k \tag{5}$$

and the momentum-energy vector $p_k=[p_{1k},p_{2k},p_{3k},p_{4k}]$, where

$$p_{lk}=\begin{cases} m_k v_{lk}, & \text{for } l=1,2,3 \\ iE_k/c, & \text{for } l=4 \end{cases} \tag{6}$$

and

$$m_k=m_0/\gamma_k \tag{7}$$

Here $E_k=m_kc^2$ denotes the energy of P at t_k, m_0 is the rest mass, and $i=\sqrt{-1}$.

As is known, the relativistic dynamical equations in the continuous case are given by (Aharoni, 1959; Bergman, 1942; Möller, 1972; Muirhead, 1973; Schwartz, 1968)

$$F_l = \frac{dp_l}{d\tau} = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} \frac{dp_l}{dt}, \qquad l = 1,2,3,4 \tag{8}$$

The replacement of derivatives in the above equations by differences yields

$$F_{lk} = \frac{1}{\gamma_k} \frac{\Delta p_{lk}}{\Delta t_k}, \qquad l = 1,2,3,4 \tag{9}$$

Finally, let us define discrete Minkowski coordinates, velocity and acceleration of P by

$$X_{lk} = \begin{cases} x_{lk}, & \text{for } l = 1,2,3 \\ ict_k, & \text{for } l = 4 \end{cases} \tag{10}$$

$$V_{lk} = \frac{\Delta X_{lk}}{\delta \tau_k} \tag{11}$$

$$A_{lk} = \frac{\Delta V_{lk}}{\delta \tau_k} \tag{12}$$

3. DISCRETE DYNAMICAL EQUATIONS IN MINKOWSKI SPACE

In this section we will give the discrete dynamical equations in Minkowski space, relying upon the definitions given in the previous section and upon the obvious properties of the operator Δ:

$$\Delta(A_k B_k) = \begin{cases} B_k \Delta A_k + A_{k+1} \Delta B_k \\ B_{k+1} \Delta A_k + A_k \Delta B_k \end{cases} \tag{13}$$

where A and B denote arbitrary quantities.

From (9), (6), and (13) we have for $l = 1,2,3$

$$F_{lk} = \frac{1}{\gamma_k} \begin{cases} \dfrac{\Delta m_k}{\Delta t_k} v_{lk} + m_{k+1} \dfrac{\Delta v_{lk}}{\Delta t_k} \\ \text{or} \\ \dfrac{\Delta m_k}{\Delta t_k} v_{l,k+1} + m_k \dfrac{\Delta v_{lk}}{\Delta t_k} \end{cases} \tag{14}$$

Let us take into account the first of the above alternatives. From (5), (2), (11), and (13) we obtain

$$F_{lk}=\frac{1}{\gamma_k}\left[\frac{\Delta m_k}{\delta\tau_k}\gamma_k^2\frac{\Delta x_{lk}}{\delta\tau_k}+\gamma_k m_{k+1}\frac{\Delta}{\delta\tau_k}\left(\gamma_k\frac{\Delta x_{lk}}{\delta\tau_k}\right)\right]$$

$$=\frac{\Delta m_k}{\delta\tau_k}\gamma_k V_{lk}+m_{k+1}\begin{cases}\dfrac{\Delta\gamma_k}{\delta\tau_k}V_{l,k+1}+\dfrac{\Delta V_{lk}}{\delta\tau_k}\gamma_k\\ \text{or}\\ \dfrac{\Delta\gamma_k}{\delta\tau_k}V_{lk}+\dfrac{\Delta V_{lk}}{\delta\tau_k}\gamma_{k+1}\end{cases}\tag{15}$$

Using (12), from (15) we have

$$F_{lk}=\gamma_k m_{k+1}A_{lk}+\frac{\Delta m_k}{\delta\tau_k}\gamma_k V_{lk}+\frac{\Delta\gamma_k}{\delta\tau_k}m_{k+1}V_{l,k+1}\tag{16}$$

$$F_{lk}=\gamma_{k+1}m_{k+1}A_{lk}+\frac{\Delta(\gamma_k m_k)}{\delta\tau_k}V_{l,k},\qquad l=1,2,3\tag{17}$$

The analogous procedure with the second alternative (14) yields

$$F_{lk}=\gamma_k m_k A_{lk}+\frac{\Delta(\gamma_k m_k)}{\delta\tau_k}V_{l,k+1}\tag{18}$$

$$F_{lk}=\gamma_{k+1}m_k A_{lk}+\frac{\Delta\gamma_k}{\delta\tau_k}m_k V_{lk}+\frac{\Delta m_k}{\delta\tau_k}\gamma_{k+1}V_{l,k+1}\tag{19}$$

Now we give the difference equations for $l=4$. From (6) and (9) we have

$$\begin{aligned}F_{4k}&=ic\Delta m_k/\delta\tau_k\\&=ic\left(-\frac{m_{k+1}}{\gamma_{k+1}}\frac{\Delta\gamma_k}{\delta\tau_k}+\frac{m_{k+1}}{\gamma_{k+1}}\frac{\Delta\gamma_k}{\delta\tau_k}+\frac{\Delta m_k}{\delta\tau_k}\right)\\&=ic\left(-\frac{\gamma_k m_{k+1}}{\gamma_k\gamma_{k+1}}\frac{\Delta\gamma_k}{\delta\tau_k}+\frac{\gamma_k}{\gamma_k}\frac{\Delta m_k}{\delta\tau_k}+\frac{m_{k+1}}{\gamma_{k+1}}\frac{\Delta\gamma_k}{\delta\tau_k}\right)\\&=-\frac{ic}{\gamma_k\gamma_{k+1}}\frac{\Delta\gamma_k}{\delta\tau_k}\gamma_k m_{k+1}+\frac{ic}{\gamma_k}\gamma_k\frac{\Delta m_k}{\delta\tau_k}\\&\quad+\frac{ic}{\gamma_{k+1}}m_{k+1}\frac{\Delta\gamma_k}{\delta\tau_k}\end{aligned}$$

Considering (12), (11), and (10), we obtain from this equation

$$F_{4k}=A_{4k}\gamma_k m_{k+1}+V_{4k}\gamma_k\frac{\Delta m_k}{\delta\tau_k}+V_{4,k+1}m_{k+1}\frac{\Delta\gamma_k}{\delta\tau_k} \tag{20}$$

But the above equation is analogous to (16). In the same way we obtain for $l=4$ equations of the form (17), (18), and (19). Therefore, taking into consideration that $m_k\gamma_k=m_0$, we may write

$$F_{lk}^1=\gamma_k m_{k+1}A_{lk}+\frac{\Delta m_k}{\delta\tau_k}\gamma_k V_{lk}+\frac{\Delta\gamma_k}{\delta\tau_k}m_{k+1}V_{l,k+1}$$

$$F_{lk}^2=m_0A_{lk}+\frac{\Delta m_0}{\delta\tau_k}V_{lk}$$

$$F_{lk}^3=m_0A_{lk}+\frac{\Delta m_0}{\delta\tau_k}V_{l,k+1}$$

$$F_{lk}^4=\gamma_{k+1}m_kA_{lk}+\frac{\Delta\gamma_k}{\delta\tau_k}m_kV_{lk}+\frac{\Delta m_k}{\delta\tau_k}\gamma_{k+1}V_{l,k+1} \tag{21}$$

On the other hand $F_{lk}^1=F_{lk}^2=F_{lk}^3=F_{lk}^4$, so that

$$F_{lk}=\sum_{i=1}^{4}\alpha_iF_{lk}^i,\qquad \text{where } \sum_{i=1}^{4}\alpha_i=1,\quad l=1,2,3,4;\ k=0,1,2,\ldots \tag{22}$$

The formula (22) presents the general discrete dynamical equation in Minkowski space obtained from (9). Let us note that for $\alpha_2=1$ and $\alpha_1=\alpha_3=\alpha_4=0$ we have

$$F_{lk}=m_0A_{lk}+(\Delta m_0/\delta\tau_k)V_{lk}$$

This formula is analogous to the continuous formula (Aharoni, 1959; Bergman, 1942)

$$F_l=m_0\frac{dV_1}{d\tau}+V_l\frac{dm_0}{d\tau} \tag{23}$$

Note moreover that for $\alpha_2=\alpha_3=\frac{1}{2}$ and $\alpha_1=\alpha_4=0$ we obtain

$$F_k^N=\frac{c^2m_k}{\left(c^2-v_k^2\right)^{1/2}\left(c^2-v_{k+1}^2\right)^{1/2}}\frac{\Delta v_k}{\Delta t_k} \tag{24}$$

D. Greenspan (1976, 1977) assumed in Minkowski space the dynamical equations of the form

$$F_{lk} = m_0 A_{lk} - \frac{\Delta m_0}{\delta \tau_k} \frac{V_{l,k+1} + V_{lk}}{2} \tag{25}$$

where the proper time is defined by

$$\delta \tau_k = \Delta t_k \left(c^2 - v_k^2\right)^{1/2}$$

The equations (25) and (24) are similar except for the sign between the terms. We see that (24) is more coherent with the continuous equation (23) than (25).

4. DISCRETE MINKOWSKI FORCE AS FOUR-VECTOR

Consider two Euclidean coordinate systems $X_1 X_2 X_3$ and $X_1' X_2' X_3'$, where $X_1' X_2' X_3'$ is in constant uniform motion with respect to $X_1 X_2 X_3$, and let us take into account two events (x_1, x_2, x_3, t) and (x_1', x_2', x_3', t'). An arbitrary vector A is called a four-vector if

$$A' = L \cdot A \tag{26}$$

where L denotes the Lorentz transformation given by

$$L = \begin{bmatrix} 1 + \beta_1^2 \frac{\varepsilon^2}{\varepsilon + 1} & \beta_1 \beta_2 \frac{\varepsilon^2}{\varepsilon + 1} & \beta_1 \beta_3 \frac{\varepsilon^2}{\varepsilon + 1} & i\beta_1 \varepsilon \\ \beta_1 \beta_2 \frac{\varepsilon^2}{\varepsilon + 1} & 1 + \beta_2^2 \frac{\varepsilon^2}{\varepsilon + 1} & \beta_2 \beta_3 \frac{\varepsilon^2}{\varepsilon + 1} & i\beta_2 \varepsilon \\ \beta_1 \beta_3 \frac{\varepsilon^2}{\varepsilon + 1} & \beta_2 \beta_3 \frac{\varepsilon^2}{\varepsilon + 1} & 1 + \beta_3^2 \frac{\varepsilon^2}{\varepsilon + 1} & i\beta_3 \varepsilon \\ -i\beta_1 \varepsilon & -i\beta_2 \varepsilon & -i\beta_3 \varepsilon & \varepsilon \end{bmatrix} \tag{27}$$

and

$$\varepsilon = \frac{1}{\left(1 - \beta^2\right)^{1/2}}, \qquad \beta^2 = \frac{u^2}{c^2}, \qquad \beta_i = \frac{u_i}{c}, \qquad i = 1, 2, 3. \tag{28}$$

In (28) u denotes the velocity of $X_1' X_2' X_3'$ with respect to the $X_1 X_2 X_3$. Note that both $V_k = [V_{1k}, V_{2k}, V_{3k}, V_{4k}]$ and $A_k = [A_{1k}, A_{2k}, A_{3k}, A_{4k}]$ are Lorentz invariant, since $\delta \tau_k$ is invariant under this transformation.

Now, let us show that the momentum-energy vector (6) is a four-vector. From (2), (7), (5), (4) we have

$$\begin{bmatrix} m'_k v'_{1k} \\ m'_k v'_{2k} \\ m'_k v'_{3k} \\ icm'_k \end{bmatrix} = \begin{bmatrix} \frac{m_0}{\gamma'_k} \frac{\Delta x'_{1k}}{\Delta t'_k} \\ \frac{m_0}{\gamma'_k} \frac{\Delta x'_{2k}}{\Delta t'_k} \\ \frac{m_0}{\gamma'_k} \frac{\Delta x'_{3k}}{\Delta t'_k} \\ \frac{m_0}{\gamma'_k} ic \end{bmatrix} = \begin{bmatrix} m_0 \frac{\Delta x'_{1k}}{\delta \tau_k} \\ m_0 \frac{\Delta x'_{2k}}{\delta \tau_k} \\ m_0 \frac{\Delta x'_{3k}}{\delta \tau_k} \\ m_0 \frac{\Delta (ict'_k)}{\delta \tau_k} \end{bmatrix}$$

$$= \begin{bmatrix} m_0 V'_{1k} \\ m_0 V'_{2k} \\ m_0 V'_{3k} \\ m_0 V'_{4k} \end{bmatrix} = L \cdot \begin{bmatrix} m_0 V_{1k} \\ m_0 V_{2k} \\ m_0 V_{3k} \\ m_0 V_{4k} \end{bmatrix}$$

$$= \cdots = L \begin{bmatrix} m_k v_{1k} \\ m_k v_{2k} \\ m_k v_{3k} \\ icm_k \end{bmatrix}$$

and the invariance is established. Hence

$$\begin{bmatrix} \Delta(m'_k v'_{1k}) \\ \Delta(m'_k v'_{2k}) \\ \Delta(m'_k v'_{3k}) \\ \Delta(icm'_k) \end{bmatrix} = L \cdot \begin{bmatrix} \Delta(m_k v_{1k}) \\ \Delta(m_k v_{2k}) \\ \Delta(m_k v_{3k}) \\ \Delta(icm_k) \end{bmatrix} \tag{29}$$

Since $\delta \tau_k$ is Lorentz invariant, the equation (29) implies at once

$$F'_k = L \cdot F_k \tag{30}$$

This means that the discrete Minkowski force is the four-vector.

5. DIFFERENCE DYNAMICAL EQUATION IN ONE-DIMENSIONAL SPACE

In continuous relativistic mechanics there exists a simple transformation from the dynamical equation (23) in Minkowski space to the classical relativistic equation (1), which can be written

$$F=\frac{d}{dt}(mv) \tag{31}$$

Denoting the Minkowski force (23) by F^M and the force (31) by F^N we have

$$F^M=\frac{1}{\left(1-v^2/c^2\right)^{1/2}}F^N$$

However, in the discrete case we cannot write, by analogy, $F_k^N=\gamma_k F_k^M$, since F_k^N could not be invariant under Lorentz transformation. So we are going to find another factor instead of γ_k. For this purpose, let us transform first the quantity $\gamma=(1-v^2/c^2)^{1/2}$:

$$\gamma=\frac{c^2\gamma}{v^2+c^2-v^2}=\frac{c^2\gamma}{v^2+c^2\gamma^2}=\frac{1}{\gamma\left(v^2/c^2\gamma^2+1\right)}$$
$$=\frac{m}{\gamma\left(mv^2/c^2\gamma^2+m\right)}=\frac{m}{\gamma\left[(dm/dv)v+m\right]}$$

or

$$\gamma=\frac{m}{\gamma d(mv)/dv} \tag{32}$$

By analogy to (32), we define the factor sought by

$$\Gamma_k=\frac{m_k}{\gamma_{k+1}\Delta(m_k v_k)/\Delta v_k} \tag{33}$$

Now we show that $\Gamma_k F_k^M=F_k^N$ and $F_k^N=F_k'^N$. From (9), (33), and (4) we have

$$\Gamma_k F_k^M=\Gamma_k\frac{1}{\gamma_k}\frac{\Delta(m_k v_k)}{\Delta t_k}$$
$$=\frac{m_k\Delta v_k}{\gamma_{k+1}\Delta(m_k v_k)}\frac{1}{\gamma_k}\frac{\Delta(m_k v_k)}{\Delta t_k}$$
$$=\frac{m_k}{\left(1-v_k^2/c^2\right)^{1/2}\left(1-v_{k+1}^2/c^2\right)^{1/2}}\frac{\Delta v_k}{\Delta t_k}$$

or

$$F_k^N = \frac{c^2 m_k}{\left(c^2 - v_k^2\right)^{1/2}\left(c^2 - v_{k+1}^2\right)^{1/2}} \frac{\Delta v_k}{\Delta t_k} \tag{34}$$

The invariance of the above equation under Lorentz transformation was proved by D. Greenspan (1976, 1977).

Note that in the limit (34) yields the classical equation (1). Finally, let us draw the formula (33) in Minkowski coordinates. For this purpose let us rewrite Γ_k as follows:

$$\Gamma_k = \frac{m_k \Delta v_k}{\gamma_{k+1} \begin{cases} \Delta m_k v_k + m_{k+1} \Delta v_k \\ \text{or} \\ v_{k+1} \Delta m_k + m_k \Delta v_k \end{cases}} \tag{35}$$

Proceeding as in Section 3 yields from (35), (2), (5), (11), (12), (13), and (21)

$$\begin{aligned} \Gamma_k^1 &= \frac{F_k^3 - \Omega_k}{\gamma_{k+1} F_k^1} \\ \Gamma_k^2 &= \frac{F_k^4 - \Omega_k}{\gamma_{k+1} F_k^2} \\ \Gamma_k^3 &= \frac{F_k^3 - \Omega_k}{\gamma_{k+1} F_k^3} \\ \Gamma_k^4 &= \frac{F_k^4 - \Omega_k}{\gamma_{k+1} F_k^4} \end{aligned} \tag{36}$$

where

$$\Omega_k = \frac{\Delta m_k}{\delta \tau_k} \gamma_{k+1} V_{k+1}$$

On the other hand $\Gamma_k^1 = \Gamma_k^2 = \Gamma_k^3 = \Gamma_k^4$; then we may write

$$\Gamma_k = \sum_{i=1}^{4} \beta_i \Gamma_k^i, \qquad \text{where } \sum_{i=1}^{4} \beta_i = 1 \tag{37}$$

The formula (37) presents the general form of the factor Γ_k in Minkowski coordinates. Hence and from (22) the equation (34) in these coordinates has the form

$$F_k^N = \left(\sum_{i=1}^{4} \beta_i \Gamma_k^i \right) \left(\sum_{i=1}^{4} \alpha_i F_k^i \right)$$

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